CONDITIONAL SIMULATION OF A COX PROCESS USING MULTIPLE SAMPLE SUPPORTS

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ABSTRACT

This paper deals with the conditional simulation of a Cox process in a family of target blocks given samples of various supports. Here it has been assumed that the distribution of the potential over all samples and blocks factorize according to the following graphical model. A node is associated to each sample and target block. Each sample node is connected to the node of the smallest sample, or block, that contains it. All block nodes are joined by edges. This graphical model possesses conditional independence relationships that can be exploited by a metropolized version of the Gibbs sampler to produce fast conditional simulations. This model can be seen as a generalisation of the discrete gaussian model traditionally used for congruent samples.

INTRODUCTION

Multiple sample support is extremely common within the mineral resource industry as different sampling campaigns are often designed with different objectives, resulting in sample data with different support sizes, shapes and configurations. Thus to incorporate all data for both estimation and uncertainty exercises is a challenging problem.

The present paper deals with the conditional simulation of a Cox process (1955) in a family of target blocks. Provided that the random intensity function, or *potential*, of the samples and the blocks satisfy some conditional independence relationships, an iterative simulation algorithm can be set up to accommodate all conditioning data. A graphical model (Lauritzen, 2001; Jordan, 2004) is introduced to specify these independence relationships in a fully consistent way. This model can be seen as a generalisation of the discrete gaussian model traditionally used in geostatistics for the non-linear estimation of local reserves starting from congruent samples (Matheron, 1976; Rivoirard, 1994; Chilès and Delfiner, 1999; Emery, 2007).

This paper starts with a summary of the salient features of the Cox process, then presents a graphical model that factorizes the joint conditional distribution of the potential of the blocks and samples, leading to a simple and fast conditional simulation algorithm of the Cox process. The relationships between this graphical model and the discrete gaussian model are then established. The proposed methodology is finally demonstrated using data emanating from a diamond placer deposit to estimate confidence limits for block concentrations.

METHODOLOGY

Presentation of the problem

A Cox process is a Poisson point process with a random intensity function, or *potential*. This potential reflects the propensity for some regions to contain more points than others. Figure 1 shows two realisations of a Cox process with their underlying potential.

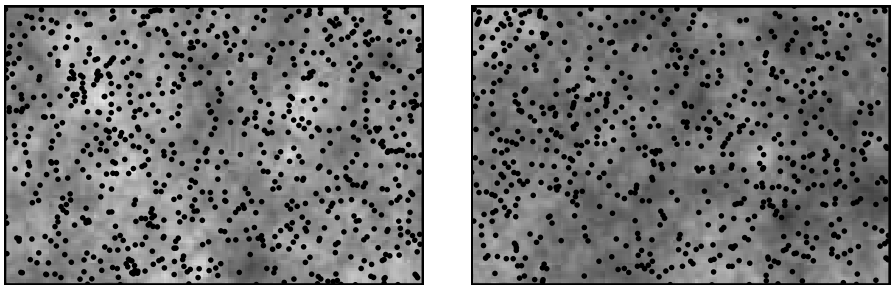

Figure 1: Two realisations of a Cox process, demonstrating the differing location of clusters, which contrasts to the standard Poisson point process.

Let $Z = (Z_x, x \in \mathbb{R}^d)$ be the random function that denotes the potential of the Cox process. The potential associated to each domain v is denoted by

$$Z_v = \int_v Z_x \, dx \qquad v \subset \mathbb{R}^d$$

The Cox process is characterized by the following conditional property. Given Z, the number of points within pairwise disjoint domains $v_1, \ldots, v_n$ are mutually independent Poisson variables with respective parameters $Z_{v_1}, \ldots, Z_{v_n}$. These does not mean that these variables are effectively independent because the potential conveys its own structure to the Cox process. For instance, the covariance between the number of points in two domains v and w is the sum of two terms

$$Cov\{N_v, N_w\} = Cov\{Z_v, Z_w\} + E\{Z_{v \cap w}\}$$

The first one is derived from the covariance of the potential whereas the second one stems from the Poisson seeding of the points.

The main concern of this paper is the conditional simulation of the number of points $(N_b, b \in B)$ in a family of pairwise disjoint target blocks given the number of points

$(N_s = n_s, s \in S)$ in a population of samples. To address this problem, it is convenient first to generate the potential of the target blocks conditional to the content[1] of the samples, and second the content of the target blocks given their potential. As the second step is straightforward (it merely amounts to simulating independent Poisson distributions), only the first step is discussed.

In principle, the first step only requires the conditional distribution of the potential of the target blocks to be considered. In practice, however, it is more advantageous to consider the joint conditional distribution of the potential of the target blocks together with the samples. The probability density function (pdf) of this distribution can be written as

$$f(z_{B\cup S} \mid n_S) \propto f(z_B)\, f(z_{S\setminus B} \mid z_B)\, p(n_S \mid z_S) \tag{1}$$

Note that this pdf is specified up to a constant, the typical situation where the Metropolis-Hasting algorithm is required for simulation. Note also that the conditional generation of the sample potential may be difficult, especially if the samples are numerous or have different supports. The graphical model that is to be introduced in the next section brings significant simplifications.

A graphical model for the potential

Let us consider a family of samples and blocks, as shown in Figure 2.

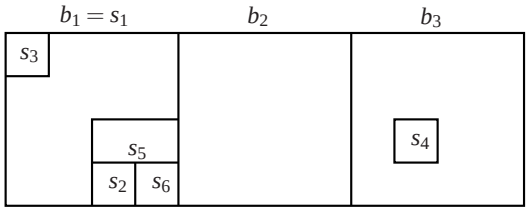

Figure 2: A family of blocks and samples. Note that s_5 includes s_2 and s_6

The mutual arrangement of these blocks and samples is fairly general, but with a few limitations. Blocks are pairwise disjoint; each sample is to be within a single block; two samples of the same block are either disjoint or ordered by inclusion; and some samples may coincide with complete blocks (e.g. production blocks).

Subject to these limitations, it is possible to represent the arrangement of blocks and samples using graph theory. A graph is specified by a set of *nodes* (or vertices) and a set of *edges* joining pairs of nodes. In our context, a node as assigned to each block and sample. each pair of block nodes is connected by an edge. Each sample node is connected to the node of the smallest sample (or block) that contains it (see Figure 3). This graph is called an *inclusion graph*.

[1] Throughout this paper, the content of a domain refers to the number of points within it.

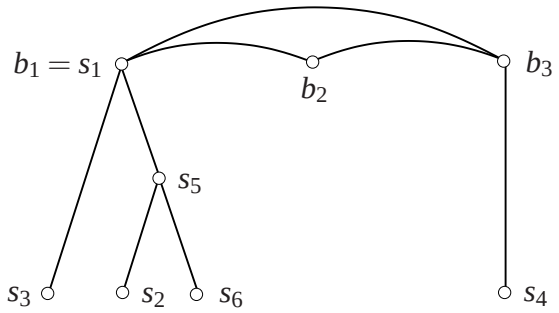

Figure 3: The inclusion graph specifying the association of blocks and samples of Figure 2

By assigning the potential of each sample and block to the node that represents it, the inclusion graph can be used to model the joint distribution of all potentials through a consistent set of conditional independence relationships. These relationships can be described as follows. The potential of samples and blocks is said to be *markovian* if for any triplet of disjoint subsets of nodes (A,B,C) such that C separates A and B[2] the potential Z_A of A and the potential Z_B of B are conditionally independent given the potential Z_C of C (Lauritzen, 2001). For example, Z_{s_4} and Z_{s_6} are conditionally independent given Z_{b_1} (as any path between s_4 and s_6 passes through b_1); however they may not be conditionally independent given Z_{b_2} (as a path between s_4 and s_6 exists that bypasses b_2, namely $s_4 - b_3 - b_1 - s_5 - s_6$).

Under this markovian assumption, the conditional distribution of the potential of the samples can be factorized. Indeed, let S_b be the family of all samples contained in block b (with possibly b itself). Note that each block node separate the nodes of its own samples from the other sample nodes, and even from the other block nodes. Accordingly, conditional independence applies and gives

$$f(z_{S\setminus B} \mid z_B) = \prod_{b\in B} f(z_{S_b\setminus b} \mid z_b) \tag{2}$$

On the other hand, the markovian assumption is not required to factorize the conditional distribution of the number of stones per sample:

$$p(n_S \mid z_S) = \prod_{b\in B} p(n_{S_b} \mid z_{S_b}) \tag{3}$$

Based on the factorizations (2) and (3), formula (1) becomes

$$f(z_{B\cup S} \mid n_S) \propto f(z_B) \prod_{b\in B} f(z_{S_b\setminus b} \mid z_b)\, p(n_{S_b} \mid z_{S_b}) \tag{4}$$

A metropolized version of the Gibbs sampler can be used to simulate the pdf specified by (4). At each iteration, a block b is selected at random. Propositional

[2]The term "C separates A and B" means that any path between a node of A and a node of B necessarily passes through a node of C.

values $\bar{z}_{S_b \cup b}$ are generated for the potential of block b and its samples. They replace the current potentials with probability

$$\alpha = \frac{p(n_{S_b} \mid \bar{z}_{S_b})}{p(n_{S_b} \mid z_{S_b})}$$

Here is the corresponding algorithm:

(i) for each $b \in B$ generate $z_b \sim f$ and $z_{S_b \setminus b} \sim f(\cdot \mid z_b)$;

(ii) select $b \sim \mathscr{U}(B)$. Generate $\bar{z}_b \sim f(\cdot \mid z_{B \setminus b})$ and $\bar{z}_{S_b \setminus b} \sim f(\cdot \mid \bar{z}_b)$;

(iii) generate $u \sim \mathscr{U}$. Put $z_{S_b} = \bar{z}_{S_b}$ if $p(n_{S_b} \mid \bar{z}_{S_b}) > u\, p(n_{S_b} \mid z_{S_b})$;

(iv) goto (ii).

There is no specific rule for generating $\bar{z}_b \sim f(\cdot \mid z_{B \setminus b})$ of step (ii). This must be designed on a case by case basis. Regarding the generation of $z_{S_b \setminus b} \sim f(\cdot \mid z_b)$ of steps (i) and (ii), it can be observed that the inclusion graph associated with block b and its samples is a tree, the root of which is precisely b. This implies the new factorization

$$f(z_{S_b \setminus b} \mid z_b) = \prod_{s \in S_b \setminus b} f(z_s \mid z_{s^*})$$

where s^* denotes the *parent support* of s. This is the smallest sample containing s, if such a sample does exist, or b itself otherwise. Accordingly, this distribution can be sequentially simulated by starting with the biggest samples and terminating with the smallest ones.

It remains to see how to compute $p(n_{S_b} \mid z_{S_b})$ of step (iii). This becomes analytically tractable when pairwise disjoint samples are considered instead of those of S_b. For each sample $s \in S_b$, let s_* be the part of s free of smaller samples, i.e. $s_* = s \setminus \cup_{s' \subsetneq s} s'$. Put $S_* = \{s_*\}$, and note that $Z_{S_b} = z_{S_b}$ if and only $Z_{S_*} = z_{S_*}$, as well as $N_{S_b} = n_{S_b}$ if and only $N_{S_*} = n_{S_*}$, z_{S_*} and n_{S_*} being defined as

$$z_{s_*} = z_s - \sum_{s' \subsetneq s} z_{s'}\, 1_{s'^* = s} \qquad n_{s_*} = n_s - \sum_{s' \subsetneq s} n_{s'}\, 1_{s'^* = s} \qquad s_* \in S_*$$

As a consequence one can write

$$p(n_{S_b} \mid z_{S_b}) = p(n_{S_*} \mid z_{S_*}) = \prod_{s_* \in S_*} p(n_{s_*} \mid z_{s_*}) = \prod_{s_* \in S_*} \exp(-z_{s_*})\, z_{s_*}^{n_{s_*}} / n_{s_*}!$$

The next section exhibits an example of a model where conditional independence properties are encountered.

A generalization of the discrete gaussian model

In this section the potential Z is assumed to be an anamorphosed standardized gaussian random function, that is

$$Z_x = \varphi(Y_x) \qquad x \in \mathbb{R}^d$$

The average potential (respectively, the average gaussian potential) over a domain v is denoted by $Z(v)$ (resp. $Y(v)$):

$$Z(v) = \frac{Z_v}{|v|} = \frac{1}{|v|}\int_v Z_x\, dx \qquad Y(v) = \frac{Y_v}{|v|} = \frac{1}{|v|}\int_v Y_x\, dx$$

Let $\dot{x}$ be a uniform point in v. The joint distribution of $\big(Y_{\dot{x}}, Y(v)\big)$ is known to be hermitian, i.e. a mixture of bigaussian distributions (Matheron, 1976). Following Emery (2007), the discrete gaussian model boils down to approximating this distribution by a bigaussian distribution. Its correlation coefficient r is positive and satisfies $r^2 = Var\{Y(v)\}$. Moreover Cartier's formula $Z(v) = E\{Z_{\dot{x}} \mid Z(v)\}$ implies

$$Z(v) = \varphi_r\left(\frac{Y(v)}{r}\right)$$

with

$$\varphi_r(y) = \int_{\mathbb{R}} \varphi(ry + \sqrt{1-r^2}u)\, g(u)\, du$$

This is shown diagramatically in Figure 4:

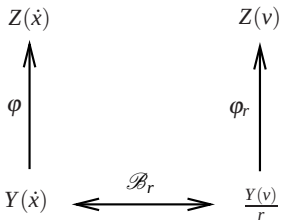

Figure 4: Construction of the discrete gaussian model. The joint distribution of $\big(Y(\dot{x}), Y(v)\big)$ is approximated by a bigaussian distribution, the correlation coefficient of which is the standard deviation of $Y(v)$

There is a way to extend this construction. The first step is to replace each sample s by a random sample $\dot{s}$, uniformly located within its parent support $\dot{s}^*$. This maintains the inclusion graph of blocks and samples. Then the following two assumptions are made:

(i) the random vector $\big(Y(v), v \in \dot{S} \cup B\big)$ is markovian;

(ii) for each sample $s \in S \backslash B$, the distribution of $\big(Y(\dot{s}), Y(\dot{s}*)\big)$ is bigaussian with correlation $\rho_{s,s^*} = r_{s^*}/r_s$, where $r_s^2 = Var\{Y(s)\}$ and $r_{s^*}^2 = Var\{Y(s*)\}$.

Note that this construction does not affect the distribution of $\big(Y(b), b \in B\big)$. It is merely aimed at approximating the multivariate distribution of $\big(Y(v), v \in \dot{S} \cup B\big)$ by a factorization according its inclusion graph:

$$g(y_{B\cup S}) = g(y_B)g(y_{S\backslash B} \mid y_B) = g(y_B)\prod_{b\in B} g(y_{S_b\backslash b} \mid y_b) = g(y_B)\prod_{\substack{b\in B\\ s\in S_b\backslash b}} g(y_s \mid y_{s^*})$$

Let v and v' be two different samples or blocks of $\dot{S} \cup B$. Starting from the previous formula and resorting to a classical result[3], it is not difficult to establish that $\big(Y(v), Y(v')\big)$ follows a bigaussian distribution. Moreover, its correlation is

$$\rho_{v,v'} = \prod_{i=1}^{n} \rho_{v_{i-1},v_i}$$

where $v = v_0, v_1, \ldots, v_n = v'$ is the shortest path between v and v' along the inclusion graph. This extends the properties of the discrete gaussian model designed for the non-linear prediction of reserves in the case of congruent samples (Chilès and Delfiner, 1999).

CASE STUDY

The marine diamond placer deposit off the West coast of Namibia has been successfully mined for the last 15 years (see Figure 5), and presents a challenging environment for grade estimation and risk appraisal.

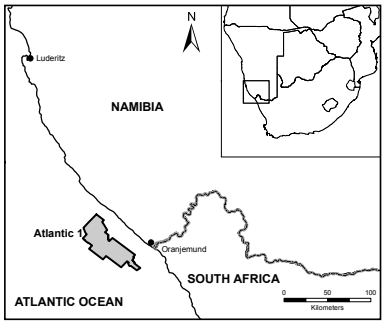

Figure 5: Geographical location of the marine diamond deposit (Atlantic 1).

As a consequence of the challenging marine environment and different geological terrains, sampling is often conducted with different sampling drills resulting in differing sample sizes. Furthermore, as the production data is collected on an individual block basis, these results can also be utilised for grade estimation providing the consequences of sample support sizes are correctly taken into account by the estimation technique. It has also become essential that meaningful confidence limits associated with block grades are established to produce achievable mine plans.

The multi-support simulation technique presented in this paper has been developed to calculate the confidence limits associated with the grade estimates. A limitation of the simulation methodology described, is that the samples must be exactly

[3]Let X, Y and Z be three standard gaussian variables. Suppose that (i) the distribution of (X,Y) is bigaussian with correlation $\rho_{X,Y}$, (ii) the distribution of (Y,Z) is bigaussian with correlation $\rho_{Y,Z}$, (iii) X and Z are conditionally independent given Y. Then the distribution of (X,Z) is bigaussian with correlation $\rho_{X,Z} = \rho_{X,Y}\rho_{Y,Z}$.

divisible into the associated block dimensions. As this is generally not the case, the compensation technique proposed by Ferreira and Lantuéjoul (2007) can be used to ensure the re-shaping and re-sizing of samples to the appropriate dimensions.

In this case study, a sample dataset comprising small single drillholes and larger production blocks of various sizes was used. The histogram of the drill samples can be fitted by a negative binomial distribution with mean 1.167 and variance 15.623. The sample variogram model consists of a nugget effect (6.536) and a spherical model with sill 9.087 and range 120m.

By establishing the relationships between the various samples using the graphical model as described above, the simulation model could be developed and numerous realisations of the placer deposit produced. The simulation was both visually and statistically validated to ensure compatibility to the conditioning data and the geological model of the deposit. Examples of typical realisations are presented in Figure 6.

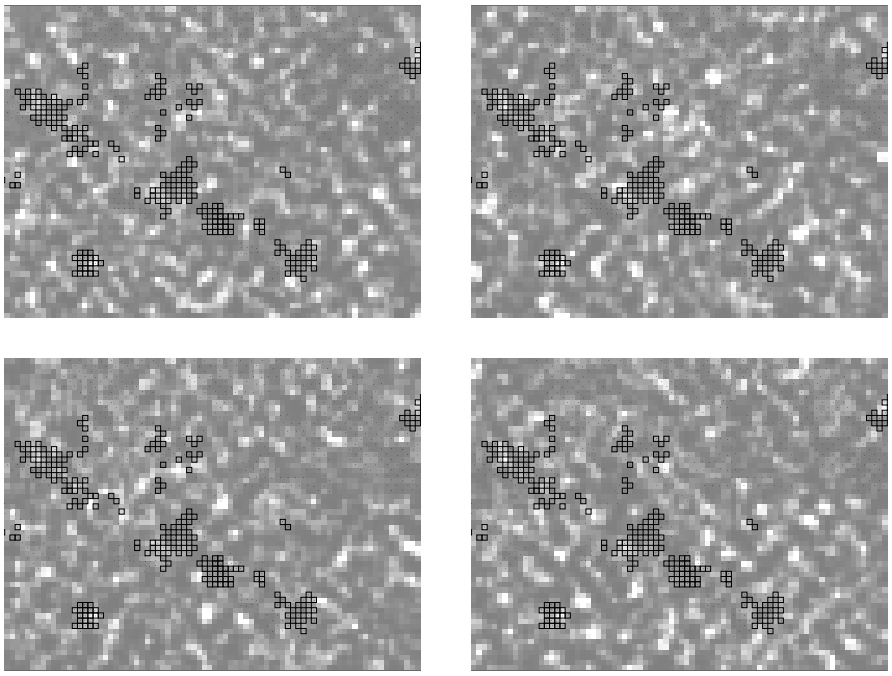

Figure 6: Four conditional simulations of the Cox process. Samples and production blocks are displayed in black.

An exercise of 100 simulations was carried out, from which the mean and standard deviation of the all the simulation realisations could be calculated (see Figure 7).

The simulation mean could also be compared the kriged estimates[4] which showed

[4]Calculated from mixed support kriging

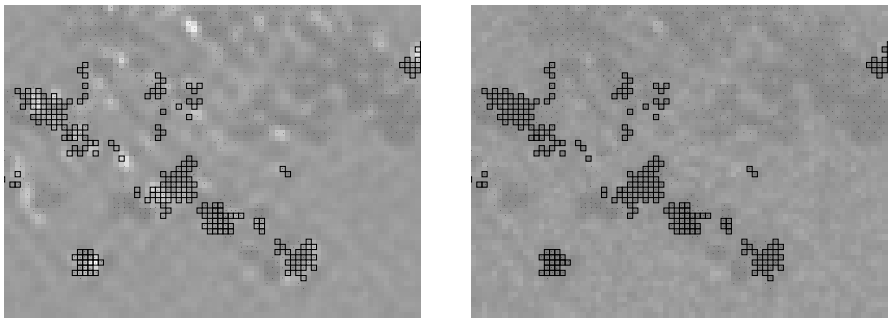

Figure 7: Mean and standard deviation computed starting from 100 conditional simulations.

a strong correlation (see Figure 8).

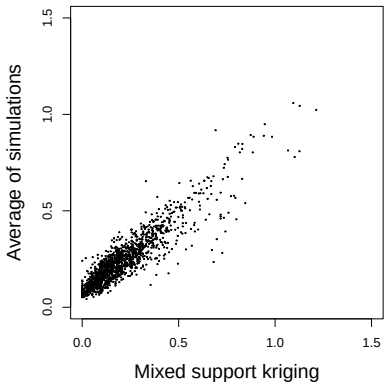

Figure 8: Scatter diagram between the mean of simulation and mixed support kriging estimates.

Confidence limits for the estimated grade have also been calculated (Figure 9) so that the relative confidence of individual blocks can be established to guide the risk profile for mine planning. An analysis of the confidence limits shows both the contribution of the sampling data and the relative contribution of the differing support sizes.

DISCUSSION

In this paper, an algorithm for the conditional simulation of the Cox process has been developed by transposing the inclusion relationships between samples and blocks in terms of a graph. Based on this graph, a consistent set of conditional independence assumptions on the potential of samples and blocks is introduced. Then the multivariate distribution of the potential factorizes, which leads to a significant simplification of its simulation algorithm (conditional or not), as well

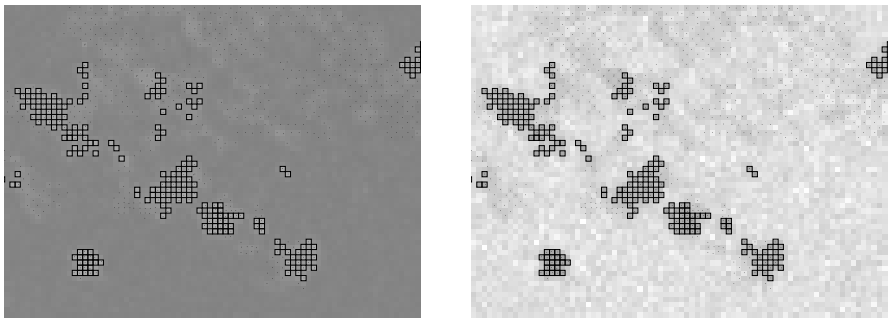

Figure 9: Quantiles 10% and 90%.

as a much faster running time.

One can question about the validity of the conditional independence assumptions made. Although it is difficult to determine their veracity, these assumptions are reasonably weak, as long as the samples are small w.r.t. the blocks. Of course, this is not true for samples that are production blocks, but such samples are treated as blocks and thereby are processed differently by the simulation algorithm. It should be pointed out that these assumptions are exactly those of the discrete gaussian model in the case where the potential is an anamorphosed gaussian random function.

Whereas the graphical model approach adopted here has been used here for the conditional simulation of a particular point process, it is beyond doubt that it has a much wider scope. It can handle many other types of mineralisation and can be applied not only to simulation but also to interpolation. It would be interesting to investigate the benefits of such an approach in the multivariate context.

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